Closing Wed: HW 13.3

Monday is a university holiday.

Exam 2 will be returned Tuesday

13.3 Areas between Curves

Example: Suppose

$MR(x) = -x^2 + 2x + 5$ dollars/item

$MC(x) = \frac{5}{2}x$ dollars/item

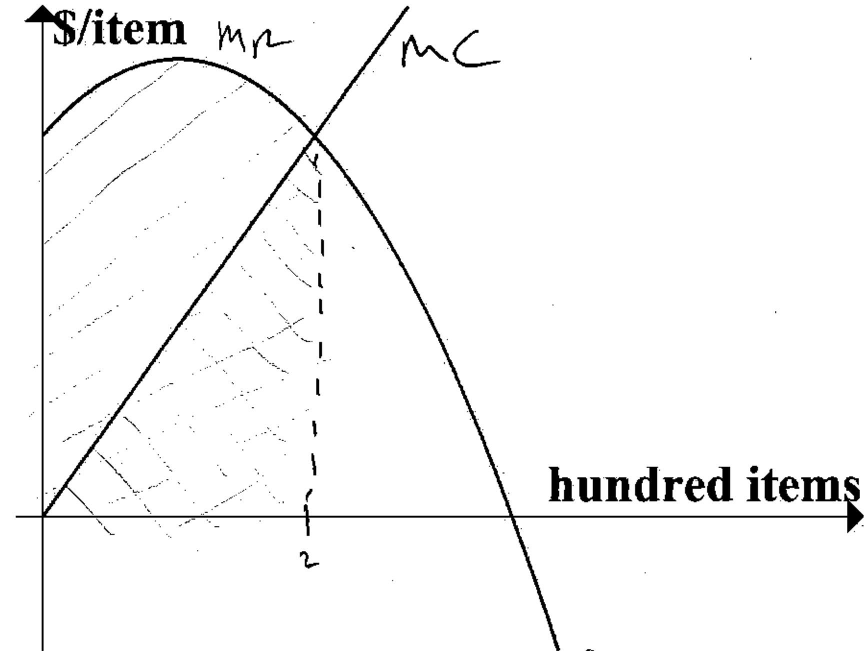
where *x* is in hundreds of items, and assume *FC* = 3 hundred dollars.

What do the following represent?

a. Area under MR from 0 to 2.

b. Area under MC from 0 to 2.

c. Area between MR & MC from 0 to 2.



NOTE: $-x^2+2x+5 \overset{?}{=} \frac{5}{2}x$ WHEN $x = 2$

(a) $\int_0^2 -x^2+2x+5\,dx = -\frac{1}{3}x^3 + x^2 + 5x \Big|_0^2 = \left(-\frac{1}{3}(2)^3 + (2)^2 + 5(2)\right) - (0) = 11.33$

$\Rightarrow TR(2) - \underbrace{TR(0)}_{0} = 11.33$ hundred dollars

(b) $\int_0^2 \frac{5}{2}x\,dx = \frac{5}{4}x^2 \Big|_0^2 = \left(\frac{5}{4}(2)^2\right) - \left(\frac{5}{4}(0)^2\right) = 5$

$\Rightarrow TC(2) - \underbrace{TC(0)}_{FC} = 5$ hundred dollars

VARIABLE COST AT 2

$TC(2) = 5 + FC$

THUS,

$TR(2) = 11.33$ hundred dollars

$VC(2) = 5$ hundred dollars

$TC(2) = 5 + FC = 5 + 3 = 8$ hundred dollars

(c) $\int_0^2 -x^2 + 2x + 5 \, dx - \int_0^2 \frac{5}{2} x \, dx$

$= \int_0^2 -x^2 + 2x + 5 - \frac{5}{2} x \, dx$ ⟸ "SHORTCUT"

$= \int_0^2 -x^2 - \frac{1}{2} x + 5 \, dx$

$= -\frac{1}{3} x^3 - \frac{1}{4} x^2 + 5x \Big|_0^2 = \left(-\frac{1}{3}(2)^3 - \frac{1}{4}(2)^2 + 5(2)\right) - (0) = 6.33$

SAME!

$TR(2) - VC(2) = 11.33 - 5 = 6.33$

ALSO

$P(2) = TR(2) - TC(2)$

$= TR(2) - (VC(2) + FC)$

$= TR(2) - VC(2) - FC \qquad = 6.33 - 3$

$\underbrace{\qquad\qquad}_{\text{AREA BETWEEN } MR \text{ \& } MC} - FC \qquad = \boxed{3.33 \text{ hundred dollars}}$

MAX PROFIT

Summary

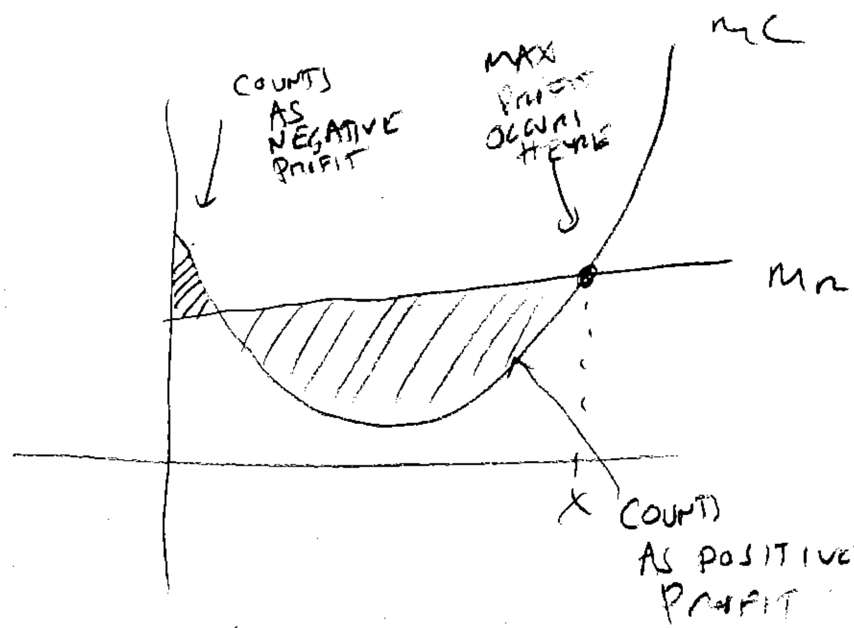
$$TR(x) = \int_0^x MR(q)dq$$

$$VC(x) = \int_0^x MC(q)dq$$

$$TC(x) = \int_0^x MC(q)dq + FC$$

$$P(x) = \int_0^x MR(q)dq - \int_0^x MC(q)dq - FC$$

$$= \int_0^x MR(q) - MC(q)\, dq - FC$$



$$P(x) = \left(\int_0^x mr(q) - mc(q)\, dq\right) - FC$$

Max profit occurs at the quantity when

$$MR(q) = MC(q)$$

Specifically, when $MR(q) > MC(q)$
switches to $MR(q) < MC(q)$

And the ***value of maximum profit*** is the *net* area from 0 to this quantity minus the fixed cost.

Example:

At time t = 0 minutes, a Red and a Green balloon are next to each other at a height of 60 feet. The ***rate of ascent*** of each balloon is given by

$$R'(t) = -\frac{1}{2}t^2 + 4t \qquad \text{feet/min}$$

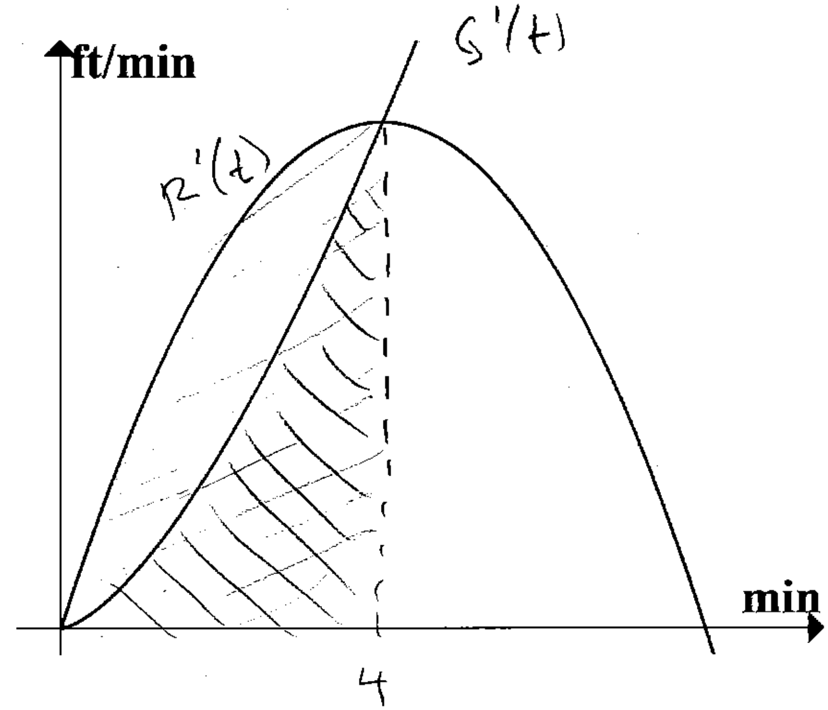
$$G'(t) = t^{3/2} \qquad \text{feet/min}$$

These graphs intersect at t = 4 minutes. What do the following represent?

a. Area under $R'(t)$ from 0 to 4.

b. Area under $G'(t)$ from 0 to 4.

c. Area between from 0 to 4.



60 feet

(a) $\int_0^4 -\frac{1}{2}t^2+4t\,dt = -\frac{1}{6}t^3+4t\Big|_0^4 = \cdots = 21.\overline{3}$ feet $= R(4) - R(0)$

60 feet

(b) $\int_0^4 t^{3/2}\,dt = \frac{2}{5}t^{5/2}\Big|_0^4 = \cdots = 12.8$ feet $= G(4) - G(0)$

$R(4) = 60 + 21.\overline{3} = 81.\overline{3}$ feet high

$G(4) = 60 + 12.8 = 72.8$ feet high

MAXIMUM DIST. BETWEEN!

(c) $\int_0^4 -\frac{1}{2}t^2+4t\,dt - \int_0^4 t^{3/2}\,dt = 21.\overline{3} - 12.8 = 8.5\overline{3}$ feet

Summary

$$R(x) = \int_0^x R'(t)dt + 60$$

$$G(x) = \int_0^x G'(t)dt + 60$$

$$R(x) - G(x) = \int_0^x R'(t)dt - \int_0^x G'(t)dt$$
$$= \int_0^x R'(t) - G'(t)\, dt$$

Maximum that the red balloon is above green balloon occurs at the time when

$$R'(t) = G'(t)$$

Specifically, when $R'(t) > G'(t)$ switches to $R'(t) < G'(t)$.

And the ***value of maximum distance between*** is the *net* area from 0 to this quantity.

In general: To find area between curves

1. Draw an accurate picture.

   Find intersections and identify

   $f(x) =$ "top function"

   $g(x) =$ "bottom function"

2. Compute:

$$\boxed{\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b f(x) - g(x)dx}$$

It gives ***change in difference between anti-derivatives*** from $x = a$ to $x = b$.

Example: Find the area of the region bounded between these curves.

$y = x^2 - 8x + 24$

$y = -x^2 + 8x.$

STEP 1

INTERSECTIONS

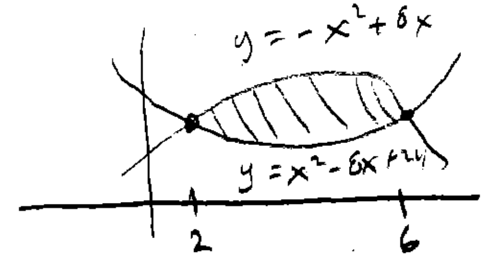
$x^2 - 8x + 24 \stackrel{?}{=} -x^2 + 8x$

$2x^2 - 16x + 24 \stackrel{?}{=} 0$

$x^2 - 8x + 12 \stackrel{?}{=} 0$

$(x-2)(x-6) \stackrel{?}{=} 0$

$x = 2$ or $x = 6$



STEP 2

WANT

$\int_2^6 -x^2 + 8x\,dx - \int_2^6 x^2 - 8x + 24\,dx$

$= \int_2^6 (-x^2 + 8x) - (x^2 - 8x + 24)\,dx$

$= \int_2^6 -2x^2 + 16x - 24\,dx$

$= -\frac{2}{3}x^3 + 8x^2 - 24x \Big|_2^6$

$= \left(-\frac{2}{3}(6)^3 + 8(6)^2 - 24(6)\right) - \left(-\frac{2}{3}(2)^3 + 8(2)^2 - 24(2)\right)$

$= \cdots = \boxed{21.\overline{3}}$ = AREA BETWEEN

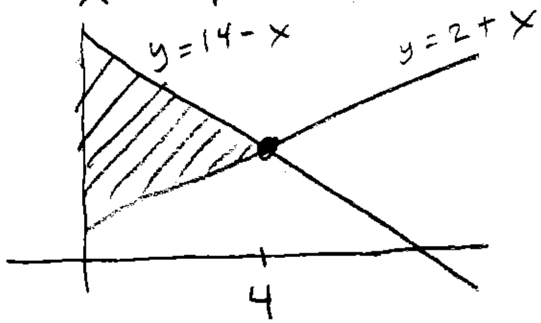
You do: Find the area of the region bounded by the y-axis and

$$y = 14 - 2x$$
$$y = 2 + x.$$

$\boxed{\text{STEP 1}}$ $14 - 2x = 2 + x$

$\Rightarrow 12 = 3x$

$\Rightarrow x = 4$



$$\int_{\boxed{0}}^{\boxed{4}} \boxed{(14 - 2x) - (2 + x)}\, dx$$

$$\int_0^4 12 - 3x \, dx$$

$$12x - \frac{3}{2}x^2 \Big|_0^4$$

$$\left(12(4) - \frac{3}{2}(4)^2\right) - 0 = 48 - 24$$

$$= \boxed{24}$$

If *x* is in hundreds of items and

$$y = MR(x) = 14 - 2x \qquad \$/\text{item}.$$
$$y = MC(x) = 2 + x \qquad \$/\text{item}.$$

What does the area you just found represent? What additional information would you like to know?

$$\int_0^4 MR(x) - MC(x)\, dx = 24$$

$$P(4) = 24 - FC$$

WOULD LIKE TO KNOW FIXED COST

MAX PROFIT!

13.4 More Integral Applications

In this section, we explore two more integral applications to business:

- Income flow
- Consumer/Supplier Surplus

Income Flow

If total income from a continuous income stream has an **annual rate** of flow given by $r(t)$,
then the total income in *k* years is

$$I(k) = \int_0^k r(t)dt.$$

This formula applies if income comes in

1. "spread out" (continuous) throughout the whole year, and
2. with an annual rate $r(t)$.

Example:

1. Constant annual rate

$$r(t) = 4000 \quad \text{dollars/year}$$

What is total income in the first 5 years

$$I(5) = \int_0^5 4000\,dt$$

$$= 4000t \Big|_0^5$$

$$= \underline{4000(5)} = \boxed{\$20,000}$$

DIDN'T REALLY NEED CALCULUS TO FIGURE THIS OUT!

2. Linearly increasing rate

$$r(t) = 3000 + 250t \quad \text{dollars/year}$$

What is total income in the first 8 years?

\$3000 in first year (spread out)
3250 in second year (spread out)
3500 in third year
etc.
⋮

$$\int_0^8 3000 + 250t \, dt$$

$$= 3000t + 125t^2 \Big|_0^8$$

$$= (3000 \cdot (8) + 125(8)^2) - (0 \quad)$$

$$= \boxed{\$32{,}000}$$

3. Exponential rate (most common, *i.e.* bank account and investments)

$$r(t) = 800e^{0.05t} \quad \text{dollars/year}$$

Aside: In this model, \$800/year is the initial rate at which income is coming ir and it is increasing at 5% per year (spread out throughout the year).

What is total income in the first 6 years

$$\int_0^6 800e^{0.05t} \, dt$$

$$\frac{800}{0.05} e^{0.05t} \Big|_0^6$$

$$16000 \, e^{0.05t} \Big|_0^6$$

$$16000 \left(e^{0.05(6)} - e^{0.05(0)}\right)$$

$$16000 \left(e^{0.3} - 1\right)$$

$$\approx \boxed{\$5597.74}$$